A survey on the missing Moore graph *

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Abstract

This is a survey on some known properties of the possible Moore graph (or graphs) Υ with degree 57 and diameter 2. Moreover, we give some new results about it, such as the following. When we consider the distance partition of Υ induced by a vertex subset C, the following graphs are distance-regular: The induced graph of the vertices at distance 1 from C when C is a set of 400 independent vertices; the induced graphs of the vertices at distance 2 from C when C is a vertex or an edge, and the line graph of Υ. Besides, Υ is an edge-distance-regular graph.

Keywords: Moore graphs, Distance-regular graph, Outindependent graph, Spectral graph theory, Automorphisms.

Mathematics Subject Classifications: 05C50; 05C25; 20B25; 20C15.

1 Introduction

The Moore bound $M(\Delta, D)$ is an upper bound on the largest possible number of vertices of a graph G with maximum degree Δ and diameter D. Trivially, if $\Delta = 1$, then $D = 1$ and $M(1,1) = 2$. If $\Delta = 2$, then $M(2, D) = 2D + 1$. For $\Delta > 2$,

$$M(\Delta, D) = 1 + \Delta \frac{(\Delta - 1)^D - 1}{\Delta - 2}.$$

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A graph whose number of vertices is the Moore bound $M(\Delta, D)$ is called a Moore graph, which is known to be necessarily regular of degree $\Delta = k$, see Singleton [21]. For diameter $D = 2$, Hoffman and Singleton [13] showed that Moore graphs only exist for $k = 2, 3, 7$ and, possibly, for $k = 57$. Here we survey the known properties of this possible Moore graph (or graphs) with degree $k = 57$, diameter $D = 2$, and order $M(57, 2) = 3250$, sometimes known as the monster (not related with the monster group), and we give some new results about it. From now on, we denote such a possible graph by Υ.

Throughout this work, we use some standard notation. Thus, Γ denotes a (simple) graph on n vertices, with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. Besides, we assume that Γ has adjacency matrix $\boldsymbol{A}$ with $d + 1$ distinct eigenvalues, and spectrum $\operatorname{sp}\Gamma = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, where $\lambda_0 (= k) > \lambda_1 > \cdots > \lambda_d$, and the superscripts stand for the multiplicities.

Some special attention is paid to distance-regularity properties. Recall that a k-regular graph Γ with diameter d is *distance-regular* if and only if there exist integers b_i and c_i, for $i = 0, ..., d$, such that for any two vertices u and v in Γ at distance i, there are exactly c_i neighbors of $v \in \Gamma_{i-1}(u)$ and b_i neighbors of $v \in \Gamma_{i+1}(u)$, where $\Gamma_i(u)$ is the set of vertices of Γ that are at distance i from u. The array of integers $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ characterizing a distance-regular graph is known as its *intersection array.* For our case of diameter $d = 2$, recall that a graph Γ is *strongly regular* if there are integers $a = k - b_1 - c_1$ and $c = c_2$ such that every two adjacent vertices have a common neighbors, and every two non-adjacent vertices have c common neighbors. For more details on distance-regular graphs and strongly regular graphs, we refer the reader to Brouwer, Cohen, and Neumaier [3], Brouwer and Haemers [5], and Van Dam, Koolen, and Tanaka [8].

1.1 Regular partitions and quotient graphs

Let $\Gamma = (V, E)$ be a graph with n vertices and adjacency matrix $\boldsymbol{A}$. A partition π of its vertex set $V = U_1 \cup U_2 \cup \cdots \cup U_m$, for $m \le n$, is called *regular* or *equitable* if the number c_{ij} of edges between a vertex $u \in U_i$ and vertices in U_j only depends on i and j. The numbers c_{ij} are usually called *intersection parameters* of the partition. The *quotient graph* of Γ with respect to π, denoted by $\pi(\Gamma)$, has as vertices the subsets U_i, for $i = 1, \ldots, m$, and c_{ij} parallel arcs from vertex U_i to vertex U_j. In this case, the *quotient matrix* $\boldsymbol{Q}$ of the partition is an $m \times m$ matrix with entries $(\boldsymbol{Q})_{ij} = c_{ij}$, and it is known that its eigenvalues constitute a subset of the set of eigenvalues of $\boldsymbol{A}$, that is, $\operatorname{ev}\boldsymbol{Q} \subset \operatorname{ev}\boldsymbol{A}$. This is the so-called Lloyd's theorem, see Godsil [11, Theorem 7.2].

1.2 Eigenvalue interlacing

Let $\boldsymbol{A}$ be a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, where $\lambda_1 \ge \cdots \ge \lambda_n$. For some $m < n$, let $\boldsymbol{S}$ be a real $n \times m$ matrix with orthogonal columns, $\boldsymbol{S}^\top \boldsymbol{S} = \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix. Then, the eigenvalues $\mu_1, \ldots, \mu_m$, where $\mu_1 \ge \cdots \ge \mu_m$, of the matrix $\boldsymbol{B} = \boldsymbol{S}^\top \boldsymbol{A}\boldsymbol{S}$ interlace the eigenvalues of $\boldsymbol{A}$. That is,

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i}, \qquad i = 1, \ldots, m. \tag{1}$$

Moreover, if the interlacing is tight, that is, for some $k \in \{0, \ldots, m\}$,

$$\lambda_i = \mu_i \quad \text{for} \quad 1 \leq i \leq k \qquad \text{and} \qquad \lambda_{n-m+i} = \mu_i \quad \text{for} \quad k+1 \leq i \leq m,$$

then $\boldsymbol{S}\boldsymbol{B} = \boldsymbol{A}\boldsymbol{S}$. In particular, if $\boldsymbol{S} = [\boldsymbol{I}\ \boldsymbol{O}]^\top$, where $\boldsymbol{O}$ is the all-0 matrix, then $\boldsymbol{B}$ is a principal submatrix of $\boldsymbol{A}$, and its eigenvalues interlace the eigenvalues of $\boldsymbol{A}$. More details about interlacing can be found in Haemers [12]. In particular, in that paper, we can find the following result.

Theorem 1.1 ([12])**.** *Let Γ be a k-regular graph, with n vertices and eigenvalues $\theta_1 (= k), \theta_2, \ldots, \theta_n$. Let Γ' be an induced subgraph of Γ, with average degree k' and n' vertices. Then*

$$\theta_n \leq \frac{nk' - n'k}{n - n'} \leq \theta_2. \tag{2}$$

Moreover, if equality is attained on either side, then Γ' is regular and so is the subgraph induced by the vertices not in Γ'.

1.3 Local spectrum of a vertex set

Consider the adjacency matrix $\boldsymbol{A}$ of a *regular* graph $\Gamma = (V, E)$ on $n = |V|$ vertices, with spectrum

$$\operatorname{sp}\Gamma := \operatorname{sp}\boldsymbol{A} = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}, \tag{3}$$

where the eigenvalues λ_i, for $0 \leq i \leq d$, are in decreasing order, and the superscripts denote the multiplicities. The set of such eigenvalues is denoted by $\operatorname{ev}\Gamma$, and their corresponding eigenspaces are $\mathcal{E}_i := \operatorname{Ker}(\boldsymbol{A} - \lambda_i \boldsymbol{I})$, for $0 \leq i \leq d$. The orthogonal projections onto the eigenspaces $\mathcal{E}_i$ are represented by the so-called (*principal*) *idempotents* of $\boldsymbol{A}$:

$$\boldsymbol{E}_i := \frac{1}{\phi_i} \prod_{\substack{j=0 \\ j \neq i}}^{d} (\boldsymbol{A} - \lambda_j \boldsymbol{I}) \qquad i = 0, \ldots, d,$$

where $\phi_i := \prod_{\substack{j=0 \\ j \neq i}}^{d} (\lambda_i - \lambda_j)$.

Given any subset C with $r := |C| \geq 1$ vertices, we consider its normalized characteristic vector $\boldsymbol{e}_C := \frac{1}{\sqrt{r}} \sum_{u \in C} \boldsymbol{e}_u$. Then, the *$C$-multiplicity* of the eigenvalue λ_i is defined by

$$m_C(\lambda_i) := \|\boldsymbol{E}_i \boldsymbol{e}_C\|^2 = \langle \boldsymbol{E}_i \boldsymbol{e}_C, \boldsymbol{e}_C \rangle = \frac{1}{r} \sum_{u,v \in C} (\boldsymbol{E}_i)_{uv} \qquad i = 0, \ldots, d. \tag{4}$$

Note that, since $\boldsymbol{e}_C$ is a unit vector, we have $\sum_{i=0}^{d} m_C(\lambda_i) = 1$. Moreover, we see that, if Γ is connected, the C-multiplicity of λ_0 is $m_C(\lambda_0) = r/n > 0$. If $\mu_0(= \lambda_0), \mu_1, \ldots, \mu_{d_C}$ (in decreasing order) represent the eigenvalues in $\operatorname{ev}\Gamma$ with non-zero C-multiplicities, the (*local*) *C-spectrum* of C is

$$\operatorname{sp} C := \{\mu_0^{\tilde{m}_0}, \mu_1^{\tilde{m}_1}, \ldots, \mu_{d_C}^{\tilde{m}_{d_C}}\}, \tag{5}$$

where $\tilde{m}_i := m_C(\mu_i)$, for $0 \leq i \leq d_C$, and $d_C(\leq d)$ is called the *dual degree* of C. It is known that, if Γ is connected, then the *eccentricity* of C, defined by $\operatorname{ecc}(C) := \max_{u\in V} \operatorname{dist}(u, C) = \max_{u\in V} \min_{v\in C} \operatorname{dist}(u, v)$, is bounded above by d_C. For more details, see Fiol and Garriga [9].

2 Known properties of Υ

We begin by recalling some basic combinatorial and spectral properties of Υ.

As it is well-known, see for example Biggs [2] or Godsil [11], Υ should be a strongly regular graph with degree $k = 57$, order $n = 57^2 + 1 = 3250$, and parameters $a = 0$, $c = 1$. Thus, Υ is a distance-regular graph with intersection array

$$\iota(\Upsilon) = \{b_0, b_1; c_1, c_2\} = \{57, 56; 1, 1\}.$$

Its different eigenvalues are $\lambda_0 = 57$, $\lambda_1 = 7$, and $\lambda_2 = -8$, which are the eigenvalues of the intersection matrix

$$\boldsymbol{B} = \begin{pmatrix} 0 & 1 & 0 \\ 57 & 0 & 1 \\ 0 & 56 & 56 \end{pmatrix}.$$

In fact, its whole spectrum turns out to be

$$\operatorname{sp}\Upsilon = \left\{57^1, 7^{1729}, -8^{1520}\right\}, \tag{6}$$

and its distance polynomials are $p_0 = 1$, $p_1 = x$, $p_2 = x^2 - 57$, giving the distance matrices $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ for $i = 0, 1, 2$.

The girth of Υ is 5. Besides, by using its geodetic properties (that is, the existence of unique paths of shortest length between vertices), the following properties are readily shown:

- Every vertex of Υ belongs to $k(k-1)^2/2 = 89376$ pentagons.
- Every pair of adjacent vertices belongs to $k(k-1) = 3192$ pentagons.
- Every path of length two belongs to 56 pentagons.
- Every path of length 3 belongs to a unique pentagon.

As a consequence of the last property, it is known that if Υ exists, then it contains an induced subgraph isomorphic to the Petersen graph minus an edge. However, as commented by Martin in [17], it is still an open problem to decide if Υ contains a Petersen graph as an induced subgraph (a problem posed by Godsil).

The known Moore graphs are vertex-transitive. Moreover, both the Petersen and the Hoffman-Singleton graphs are not Cayley graphs. The study of the automorphism group of the missing Moore graph, $G = \mathrm{Aut}(\Upsilon)$, was initiated by Aschbacher [1], by proving that G cannot be a rank-3 group (and, hence, Υ cannot be distance-transitive). In the 1960's Higman, in one of his (unpublished) lectures, improved Aschbacher's result by showing that Υ cannot be vertex-transitive (see Cameron [7]). Closer research on the structure of G was first done by Makhnev and Paduchikh [16], who proved that $|G| \leq 550$ if G has even order; and later by Mačaj and Širáň [15], who proved that the order of G is bounded above by 375 if the order is odd, and by 110 if the order is even. It is worth noting that, in both cases, the bound is very small compared to the order of Υ. For more details, see Miller and Širáň [18].

Another property of our possible graph, proved by Sabidussi [19], is that Υ would be the antipodal quotient of an antipodal graph with odd diameter and extremal girth (that is, the girth is not less than the diameter).

2.1 The spectra of induced subgraphs

Let us see how the interlacing results of Subsection 1.2 give valuable information about the spectra of some induced subgraphs of Υ.

Lemma 2.1. *Let $\Gamma = (V, E)$ be an induced k-regular subgraph of Υ. If $m = |V| > 1730$ and $k > 7$, then Γ is connected and its eigenvalues are*

$$(57 \geq)\, k > 7^{m-1521} \geq \theta_1 \geq \cdots \geq \theta_{3250-m} \geq -8^{m-1730}. \tag{7}$$

Proof. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ be, respectively, the eigenvalues of Υ (given in (6)) and of Γ. Since Γ is regular, $\mu_1 = k$. Besides, as the adjacency matrix of Γ is a principal submatrix of that of Υ, the eigenvalues of Γ interlace those of Υ. In particular, for $i = 2, \ldots, m - 1520$, we have

$$7 = \lambda_i \geq \mu_i \geq \lambda_{n-m+i} = 7.$$

Thus, the eigenvalue 7 must appear in the spectrum of Γ at least with multiplicity $m-1521$ (notice that, for $i = m-1520$, $\lambda_{n-m+i} = \lambda_{1730} = 7$). In the same way, by using interlacing, we prove that the eigenvalue -8 must appear in (7) at least with multiplicity $m - 1730$. As all the multiplicities sum up to m, we must have $3250 - m$ (different or not) additional eigenvalues θ_i. Finally, as $k > 7$, k has multiplicity one, which for a regular graph implies being connected. $\square$

3 Outindependent graphs

In this section, we describe some results about the spectra of the possible outindependent subgraphs of Υ. The concept of outindependent graph was introduced by Fiol and Garriga [10] as follows. Let C be an independent vertex r-set of a strongly regular graph Γ. Then, C induces the partition $V = C_0 \cup C_1$, where $C_0 := C$ and $C_1 = V \setminus C$. The graph $\Gamma_C = (C_1, E_1)$, induced by the vertices that are not in C (that is, that are in C_1), is called the *outindependent graph*.

In the case of Υ, the well-known Hoffman-Lovász's upper bound, hereafter denoted by $\textsc{h}$, for the independence number of a regular graph (see Haemers [12], and Lovász [14]) yields

$$\alpha \le \textsc{h} = \frac{n}{1 - \frac{\lambda_0}{\lambda_2}} = 400. \tag{8}$$

Alternatively, as Υ has diameter two, α equals the maximum number $r = r_2$ of vertices mutually at distance two, which yields now $r \le \textsc{h} = 400$.

The following result gives the spectrum of Υ_C in terms of $r = |C|$.

Theorem 3.1 ([10]). *Let C be an independent r-set of Υ. Then,*

(*a*) *If $1 \le r < 400$, then the outindependent graph Υ_C is non-regular and has spectrum*

$$\operatorname{sp} \Upsilon_C = \{\alpha^1, 7^{1729-r}, \beta, -1^{r-1}, -8^{1520-r}\},$$

where $\alpha = 28 + \sqrt{841 - r} \in (49, 57)$ and $\beta = 28 - \sqrt{841 - r} \in (-1, 7)$.

(*b*) *If $r = 400$, then the outindependent graph Υ_C is regular and has spectrum*

$$\operatorname{sp} \Upsilon_C = \{49^1, 7^{1330}, -1^{399}, -8^{1120}\}.$$

Notice that the case (b) corresponds to the case (a) with $\alpha = 49$ and $\beta = 7$. Apart from this, other integer eigenvalues appear. More precisely, when $r = s(58 - s)$, we get $\alpha = 57 - s$ and $\beta = s - 1$, for $s = 1, \ldots, 8$.

The above spectra were also found by Schwenk [20], when C is a vertex- or edge-neighborhood. These correspond to the cases $r = 57, 112$ ($s = 1, 2$) which, as shown in the following result, are the only situations where the outindependent graph is not connected.

Proposition 3.2 ([10]). *Let C, with $r = |C|$, be an independent set of $\Upsilon = (V, E)$. Then, the outindependent graph Υ_C is connected and with diameter $3 \le D \le 4$, except for the cases $C = \Gamma(u)$ or $C = \Gamma(uv)$ (where $u \in V$ and $uv \in E$) in which Υ_C has two components.*

4 Completely regular codes in Υ

In this section, we study some bounds on the maximum number of independent copies (that is, without edges between them) of some subgraphs in Υ.

Recall that, the *inner distribution* of a subset $C \subset V$ is given by the numbers $r_0, \ldots, r_d$, where r_i is the mean number of vertices in C at distance i (in the graph Γ) from a vertex $u \in C$. That is,

$$r_i = \frac{1}{|C|} \sum_{u \in C} |\Gamma_i(u) \cap C| = \sum_{i=0}^{d} m_C(\lambda_j) p_i(\lambda_j),$$

for $0 \le i \le d$. As commented by Godsil [11], these numbers determine the probability that a randomly chosen pair of vertices from C are at distance i. In [9], Fiol and Garriga gave the following result about how to compute the local multiplicities of a vertex subset C in a distance-regular graph by using the inner distribution of C.

Proposition 4.1 ([9]). *Let Γ be a distance-regular graph with n vertices, diameter d, spectrum $\operatorname{sp}\Gamma = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$ and distance polynomials $\{p_i\}_{0 \le i \le d}$. Let C be a vertex subset of r vertices and inner distribution r_i, for $0 \le i \le d$. Then, the C-multiplicities are given by*

$$m_C(\lambda_j) = \frac{m_j}{n} \sum_{i=0}^{d} r_i \frac{p_i(\lambda_j)}{p_i(\lambda_0)} \qquad 0 \le j \le d.$$

Then, by imposing the non-negativity of the C-multiplicities, we have the following result (the cases (a), (c), and (d) can also be found in Fiol and Garriga [9], whereas the case (e) is mentioned without proof in Mačaj and Širáň [15]).

Proposition 4.2. *Let C be a set of m independent equal subgraphs Γ (that is, without edges between them) of Υ. Then,*

- (a) *If $\Gamma = K_1$ (a singleton), then $m = |C| \le 400$.*
- (b) *If $\Gamma = K_2$, then $m \le 225$.*
- (c) *If $\Gamma = C_5$ (a pentagon), then $m \le 100$.*
- (d) *If $\Gamma = P$ (the Petersen graph), then $m \le 55$.*
- (e) *If $\Gamma = HS$ (the Hoffman-Singleton graph), then $m \le 15$.*

Moreover, in all the cases, if the bound is attained then C is a completely regular code of Υ.

Proof. (*a*) $C = \{m \text{ independent vertices}\}$: $|V(C)| = m$, $r_0 = 1$ and $r_1 = 0$. Then,

$$\begin{aligned}
m_C(\lambda_0) &= \frac{1}{3250}\left(\frac{1}{1} + r_2\frac{3192}{3192}\right), \\
m_C(\lambda_1) &= \frac{1729}{3250}\left(\frac{1}{1} - r_2\frac{8}{3192}\right), \\
m_C(\lambda_2) &= \frac{1520}{3250}\left(\frac{1}{1} + r_2\frac{7}{3192}\right).
\end{aligned}$$

These values satisfy $\sum_{j=0}^{2} m_C(\lambda_j) = 1$, as expected and, from $m_C(\lambda_i) \geq 0$, we get $r_2 \leq 399$, which implies $r = r_0 + r_1 + r_2 \leq 400$ and $m \leq \frac{r}{1} = 400$. Moreover, if $\boldsymbol{r} = (1, 0, 399)$, then

$$\boldsymbol{m}_C = (m_C(\lambda_0), m_C(\lambda_1), m_C(\lambda_2)) = \left(\tfrac{8}{65}, 0, \tfrac{57}{65}\right).$$

Then, since there are only two nonzero C-multiplicities, the distance partition induced by C is an equitable partition or, equivalently, C is a completely regular code with covering radius (or eccentricity) 2 (see Fiol and Garriga [9]). The proofs of the other cases are similar and we only list the obtained parameters.

(*b*) $C = \{m \text{ independent edges}\}$: $|V(C)| = 2m$, and $r_0 = r_1 = 1$. We get $r_2 \leq 448$, which implies $r = r_0 + r_1 + r_2 \leq 450$ and $m \leq \frac{r}{2} = 225$. If $\boldsymbol{r} = (1, 1, 450)$, then $\boldsymbol{m}_C = \left(\frac{9}{65}, 0, \frac{56}{65}\right)$.

(*c*) $C = \{m \text{ independent pentagons}\}$: $|V(C)| = 5m$, $r_0 = 1$ and $r_1 = 2$. We get $r_2 \leq 497$, which implies $r = r_0 + r_1 + r_2 \leq 500$ and $m \leq \frac{r}{5} = 100$. If $\boldsymbol{r} = (1, 2, 497)$, then $\boldsymbol{m}_C = \left(\frac{7}{13}, 0, \frac{11}{13}\right)$.

(*d*) $C = \{m \text{ independent Petersen graphs}\}$: $|V(C)| = 10m$, $r_0 = 1$ and $r_1 = 3$. We get $r_2 \leq 546$, which implies $r = r_0 + r_1 + r_2 \leq 550$ and $m \leq \frac{r}{10} = 55$. If $\boldsymbol{r} = (1, 3, 546)$, then $\boldsymbol{m}_C = \left(\frac{11}{65}, 0, \frac{59}{65}\right)$.

(*e*) $C = \{m \text{ independent Hoffman-Singleton graphs}\}$: $|V(C)| = 50m$, $r_0 = 1$ and $r_1 = 7$. We get $r_2 \leq 742$, which implies $r = r_0 + r_1 + r_2 \leq 750$ and $m \leq \frac{r}{50} = 15$. If $\boldsymbol{r} = (1, 7, 742)$, then $\boldsymbol{m}_C = \left(\frac{3}{13}, 0, \frac{10}{13}\right)$.

□

The corresponding intersection diagrams of the cases of Proposition 4.2 are depicted in Figure 1.

In fact, if we do not mind about the local spectra of the vertex sets, the results of Proposition 4.2 can be proved by using interlacing. Indeed, if we apply Theorem 1.1 with $n = 3250$, $k = 57$, and $\theta_3 = -8$, the first inequality of (2) yields that any k'-regular induced subgraph of Υ has order satisfying $n' \leq 50(8 + k')$. Then, with $k' \in \{0, 1, 2, 3, 7\}$, the results of Proposition 4.2 follow.

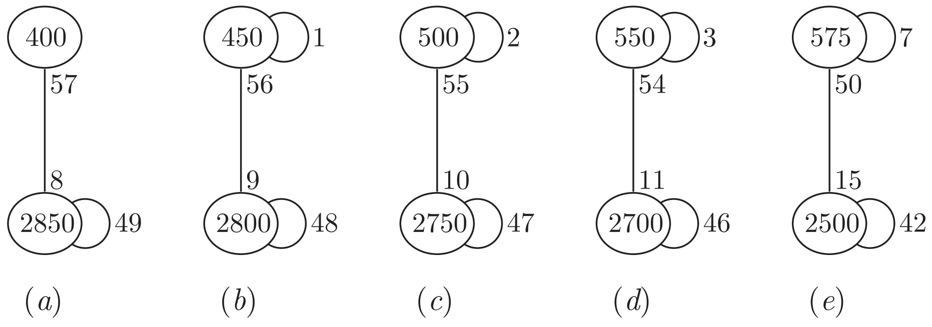

Figure 1: The intersection diagrams with different equitable partitions given by sets of independent subgraphs, in the cases in which these subgraphs are: (a) vertices, (b) edges, (c) pentagons, (d) Petersen graphs, and (e) Hoffman-Singleton graphs.

5 Distance-regularity properties

In this section, we show that the existence of Υ would imply also the existence of some edge-distance-regular and distance-regular graphs (usually, as induced subgraphs of Υ). Unfortunately, up to now nothing is known about the existence or non-existence of such distance-regular graphs.

First, recall that a graph is *edge-distance-regular* when the distance partition induced by each edge is equitable, and with the same intersection numbers independently of the edge (see Cámara, Dalfó, Delorme, Fiol, and Suzuki [6]). A *generalized odd graph* is a distance-regular graph with diameter d and odd-girth (that is, the shortest cycle of odd length) $2d+1$. In [6] it is was proved that a graph is edge-distance-regular if and only if it is a bipartite distance-regular graph or a generalized odd graph. As a consequence, we show the first part of the next theorem, whereas the second part is due to Brouwer, Cohen, and Neumaier [3, Theorem 4.2.16].

Theorem 5.1. (a) *The graph Υ is an edge-distance-regular graph with intersection array*

$$\iota(\Upsilon) = \{56, 56; 1, 2\}.$$

(b) *The line graph $L(\Upsilon)$ is a distance-regular graph with intersection array*

$$\iota(L(\Upsilon)) = \{112, 56, 55; 1, 1, 4\}.$$

In the framework of Proposition 3.2, the next result is well-known (see, for instance, Fiol and Garriga [10]), and it states that the graphs induced by the vertices at distance two from either a vertex or an edge, are also distance-regular. See the corresponding intersection diagrams of Υ in Figure 3 $(a1)$ and $(a2)$.

Theorem 5.2. *(a) For any vertex u in Υ, let $\Upsilon(u)$ be the graph induced by the vertices at distance two from u. Then, $\Upsilon(u)$ is a distance-regular graph with diameter three, intersection array*

$$\iota(\Upsilon(u)) = \{56, 55, 1; 1, 1, 56\},$$

and spectrum

$$\operatorname{sp}\Upsilon(u) = \{56^1, 7^{1672}, -1^{56}, -8^{1463}\}.$$

(b) For any edge $uv \in E$, let $\Upsilon(uv)$ be the graph induced by the vertices at distance two from uv. Then, $\Upsilon(uv)$ is a distance-regular graph with diameter three, intersection array

$$\iota(\Upsilon(uv)) = \{55, 54, 2; 1, 1, 54\},$$

and spectrum

$$\operatorname{sp}\Upsilon(uv) = \{55^1, 7^{1617}, -1^{110}, -8^{1408}\}.$$

In fact, the graphs of the above result are equivalent to Υ in the following sense. In case (a) the graph $\Upsilon(u)$ would be an antipodal 56-cover of the complete graph K_{57}, and then we could add $1 + 57$ new vertices to get the Moore graph back. Similarly, in case (b), the graph $\Upsilon(uv)$ has eigenvalue -1 and, hence, its distance-3 graph $\Upsilon(uv)_3$ would be strongly regular (see, for instance, Brouwer and Fiol [4]). Namely, $\Upsilon(uv)_3$ should be the 56×56 grid (that is, the line graph of $K_{56,56}$). Then, from $\Upsilon(uv)$ we could add $2(1 + 56)$ vertices and reconstruct the Moore graph. The intersection diagrams of $\Upsilon(u)$ and $\Upsilon(uv)$ are represented in Figure 2.

In the following result, we present the possible regular partitions of Υ obtained as distance partitions induced by a Moore graph with diameter two.

Proposition 5.3. *If Υ contains an induced $(k, 2)$-Moore graph Γ with $k = 2, 3, 7$, then the only possible distance partitions induced by $V(\Gamma)$ are those of Figure 3 $(a3)$–$(a5)$.*

Proof. Since Γ has order $k^2 + 1$ and Υ has diameter 2, the distance partition induced by $V(\Gamma)$ should have the intersection diagram of Figure 3 (a), where x, y are integers to be determined. Then, the corresponding quotient matrix would be

$$\boldsymbol{B} = \begin{pmatrix} k & 57-k & 0 \\ 1 & 56-x & x \\ 0 & y & 57-y \end{pmatrix},$$

with eigenvalues

$$\mu_0 = 57, \qquad \mu_{1,2} = \frac{1}{2}\left(k - x - y + 56 + \sqrt{(k + x + y - 58)^2 + 4x}\right),$$

where, due to the intersection diagram, it must be $y = \frac{(k^2+1)(57-k)x}{3250-(k^2+1)(58-k)}$. Therefore, the result follows from Lloyd's theorem (see Subsection 1.1) since, for $k = 2, 3, 7$, the only possible values of x giving $\{\mu_1, \mu_2\} \subset \{7, -8\}$ (in fact $\{\mu_1, \mu_2\} = \{7, -8\}$) are, respectively, $x = 54, 50, 14$, as shown in Figure 3 $(a3)$–$(a5)$. □

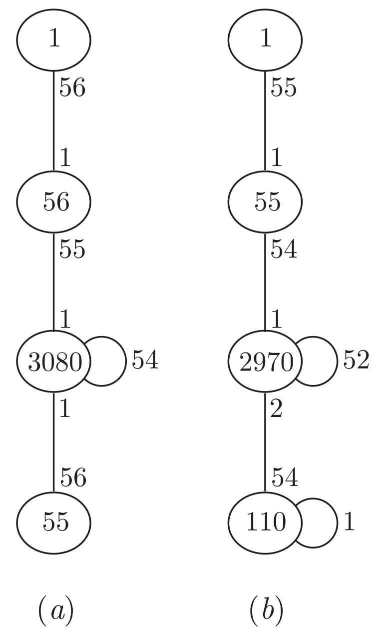

Figure 2: (a) The intersection diagram of $\Upsilon(u)$. (b) The intersection diagram of $\Upsilon(uv)$.

Notice that the above reasoning also applies for the already considered cases of a vertex $(k = 0)$ and an edge $(k = 1)$ shown in Figure 3 $(a1)$ and $(a2)$. However, in contrast with these cases, the graphs induced by the vertices at distance two (respectively, one) from a pentagon $\Upsilon(C_5)$ or a Petersen graph $\Upsilon(P)$, (respectively a Hoffman-Singleton graph $\Upsilon(HS)$), if there exist, seem to have a much more involved structure than $\Upsilon(u)$ and $\Upsilon(uv)$.

Proposition 5.4. *Let $\Upsilon(\Gamma)$ be the subgraph of Υ induced by the vertices at distance two of the regular partitions induced by C_5 or P, or the graph induced by the vertices at distance one from HS. Then, $\Upsilon(\Gamma)$ has at least $d + 1 = 5$ different eigenvalues.*

Proof. First notice that, by Proposition 5.3, $\Upsilon(\Gamma)$ has order $m \in \{2970, 2700, 2500\}$ and it is regular with degree $k \in \{52, 47, 42\}$. Thus, as $m > 1730$, we can assume that, if $d \leq 4$, then $\Upsilon(\Gamma)$ has spectrum $\{k^1, 7^{m_1}, \sigma^{m_2}, \tau^{m_3}, -8^{m_4}\}$, where σ or τ could be not necessarily different from 7 or -8, and $m_4 < 1520$. Thus, as Υ has girth 5, this must also be the case for $\Upsilon(\Gamma)$. This leads to the following system by considering the values of $\operatorname{tr} \boldsymbol{A}^\ell$, where tr is the trace of the corresponding matrix, and $\boldsymbol{A}$ is the adjacency matrix of $\Upsilon(\Gamma)$, for

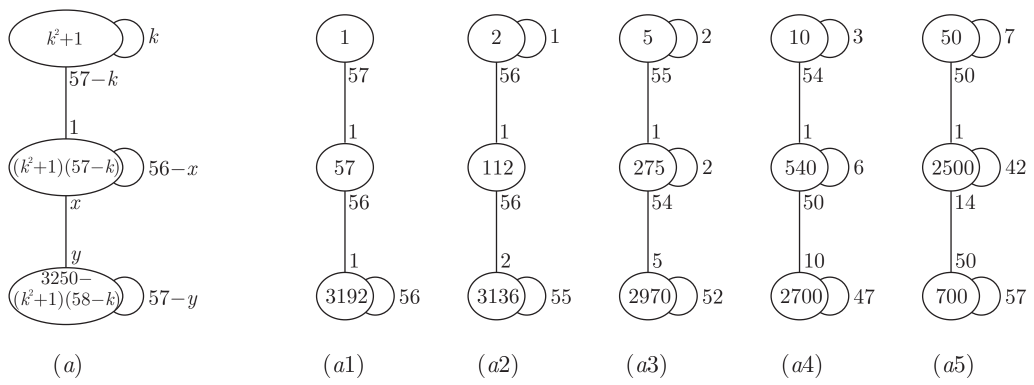

Figure 3: The intersection diagrams when Υ is hanging from one of the following Moore graphs: (a) the general case, $(a1)$ a vertex, $(a2)$ an edge, $(a3)$ a pentagon, $(a4)$ a Petersen graph, and $(a5)$ a Hoffman-Singleton graph.

$\ell = 0, \ldots, 4$,

$$\begin{aligned}
1 + m_1 + m_2 + m_3 + m_4 &= m,\\
k + 7m_1 + m_2\sigma + m_3\tau - 8m_4 &= 0,\\
k^2 + 7^2m_1 + m_2\sigma^2 + m_3\tau^2 + 8^2m_4 &= mk,\\
k^3 + 7^3m_1 + m_2\sigma^3 + m_3\tau^3 - 8^3m_4 &= 0,\\
k^3 + 7^3m_1 + m_2\sigma^3 + m_3\tau^3 - 8^3m_4 &= mk(2k-1).
\end{aligned}$$

Solving, for instance for σ, τ, m_1, m_2, m_3, we obtain all these values in terms of m_4. But, in all cases, a computer search for $m_4 < 1520$ yields non-integer values of the other multiplicities, against the hypothesis. □

Looking back to the results of Proposition 4.2, we can also wonder if some of the subgraphs of Υ, induced by the vertices at distance one from a maximum set of m independent subgraphs Γ, is distance-regular. In this context, an affirmative answer was given in Fiol and Garriga [10] when Γ is a singleton, with $m = |C| = 400$.

Theorem 5.5 ([10])**.** *When C attains the Hoffman-Lovász's bound $\mathrm{H} = 400$, the outindependent subgraph Υ_C of Υ is a distance-regular graph with order $n = 2850$, degree $k = 49$, diameter $D = 3$, intersection array*

$$\iota(\Upsilon_C) = \{49, 48, 8; 1, 1, 42\},$$

and spectrum given by Theorem 3.1(b).

Curiously enough, the possible outindependent graph Υ_C seems to inherit some properties of Υ, as shown in the following result. See the intersection diagram of Υ_C in Figure 4.

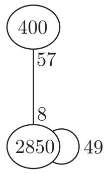

Figure 4: The intersection diagram of Υ_C when Υ is hanging from 400 vertices.

Proposition 5.6. *Let C' be a maximum set of m independent vertices in the graph Υ_C of Theorem 5.5 of Υ. Then, C' is a completely regular code of Υ_C, with $m = 400$ vertices, covering radius 2, inner distribution $(r_0, r_1, r_2, r_3) = (1, 0, 343, 56)$, and C'-local multiplicities*

$$\boldsymbol{m}_C = (m_C(\lambda_0), m_C(\lambda_1), m_C(\lambda_2), m_C(\lambda_3)) = \left(\tfrac{8}{57}, 0, 0, \tfrac{49}{57}\right).$$

Proof. The result is obtained reasoning as in the proof of Proposition 4.2, and taking into account that the distance polynomials of Υ_C are $p_0 = 1$, $p_1 = x$, $p_2 = x^2 - 49$, and $p_3 = \frac{1}{42}(x^3 - 40x^2 - 97x + 1960)$. □

Finally, we mention that the same negative results of Proposition 5.4 are obtained for the other outindependent subgraphs of Υ appeared in Proposition 4.2 (see Figure 1 (b)–(e).

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